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JWEET

Comparison of LQR and Pole Placement Design Controllers for Controlling the Inverted Pendulum

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Abstract – *In this paper an inverted pendulum is modeled firstly by using Euler – Lagrange energy equation for stabilization of the pendulum. To control the modeled system, both full-state feedback and Linear Quadratic Regulator controller methods are applied and the results are compared. After that, a pre-compensator is implemented to eliminate the steady-state error. Linear Quadratic Regulator is an optimal technique of pole placement method which defines the optimal pole location based on a definite cost function. The investigated system develops classical inverted pendulum by forming two moving masses. The motion of two masses in the pendulum which slide along the horizontal plane is controllable.*
Keywords: *State feedback control, Linear Quadratic Regulator control, Inverted pendulum, Mathematical modeling, pre-compensator*

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INTRODUCTION

Inverted pendulum systems by moving a cart along a horizontal track are one of the most popular benchmarks to demonstrate the control techniques [1]. They are unstable and nonlinear systems and can be useful to illustrate the concepts in linear control such as the stabilization of unstable systems and also in demonstrating some of the ideas in nonlinear control. Instability of the inverted pendulum makes it in the case that it may fall over any time in any direction unless a suitable control force is implemented [2].

Inverted pendulum systems are highly nonlinear, but they can be easily controlled by using linear control techniques in an almost vertical position. If the system is controllable, this method gives excellent stability margins [3, 4]. In this paper, we will employ two renowned steady state methods (FSF and LQR) to control the system and a comparison between these two controllers will show the proper controller for the inverted pendulum. The guaranteed margins in LQR design are 60 degree phase margin, infinite gain margin, and -6dB gain reduction margin.

MODELING OF INVERTED PENDULUM:

The inverted pendulum is one of the classical problems in the control theory which is a tical benchmark for testing control algorithms; the pendulum is connected to the pivot on top of the cart. Schematic of the inverted pendulum is shown in the Figure below:

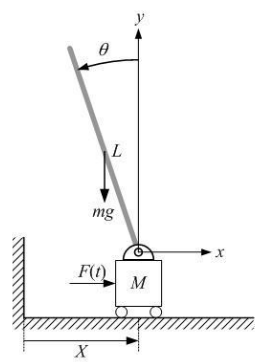

Fig.1- Inverted pendulum.

A free body diagram of the system is shown in Fig.2, and characterizes the forces and moments with implementing Newton's second law to the motor cart yields.

$$m_c \ddot{p} = F - N \tag{1}$$

By applying Newton's second law to the inverted pendulum in both (horizontal and vertical) directions.

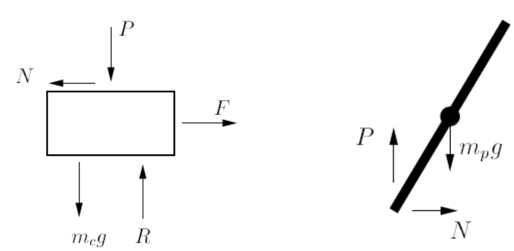

Fig.2- Free Body Diagram for Cart/Pendulum System

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We achieve:

$$N = m_p \frac{d^2}{dt^2}(p + l\sin(\theta)) \tag{2}$$
$$= m_p \frac{d^2}{dt^2}(\ddot{p} + l\cos(\theta)\ddot{\theta} - l\sin(\theta)(\dot{\theta})^2)$$
$$P - m_p g = m_p \frac{d^2}{dt^2}(l\cos(\theta))$$
$$= m_p g + m_p l(-\sin(\theta)\ddot{\theta} - \cos(\theta)(\dot{\theta})^2)$$

Pendulum's center of mass is balanced in the moments as:

$$I\ddot{\theta} = Pl\sin(\theta) - Nl\cos(\theta) \tag{3}$$

Substituting Eq. 2 into Eq. 1 we achieve:

$$F = (m_c + m_p)\ddot{p} + m_p l\cos(\theta)\ddot{\theta} - m_p l\sin(\theta)(\dot{\theta})^2 \tag{4}$$

By substituting Eq. 2 and 3 into Eq.4 yields:

$$\ddot{\theta}(I + m_p l^2) = m_p gl\sin(\theta) - m_p l\ddot{p}\cos(\theta) \tag{5}$$

To simplify the equation above:

$$M = m_c + m_p \tag{6}$$
$$L = \frac{I + m_p l^2}{m_p l}$$

Relationship between force and voltage for the motor cart can be written as:

$$F = \frac{K_m K_g}{Rr}V - \frac{K_m^2 K_g^2}{Rr^2}\dot{p} \tag{7}$$

Omitting $\ddot{\theta}$ from Eq.5, $\ddot{p}$ from Eq. 6, and replacing M, L, and the equation for force yields

$$\ddot{p}(M - \frac{m_p l\cos^2(\theta)}{L}) = \frac{K_m K_g}{Rr}V - \frac{K_m^2 K_g^2}{Rr^2}\dot{p} \tag{8}$$
$$-\frac{m_p l_g}{L}\cos(\theta)\sin(\theta) + m_p l\sin(\theta)(\dot{\theta})^2$$
$$\ddot{\theta}(L - \frac{m_p l\cos^2(\theta)}{M}) = g\sin(\theta) \tag{9}$$
$$-\frac{m_p l(\dot{\theta})^2}{M}\cos(\theta)\sin(\theta)$$
$$-\frac{\cos(\theta)}{M}(\frac{K_m K_g}{Rr}V - \frac{K_m^2 K_g^2}{Rr^2}\dot{p})$$

Let us describe the state-vector as below:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{10}$$

The system is now modeled. In this step, for simplify the equation, it is linearized about the equilibrium [0 0 0 0]T. Note that $\theta = 0$ represents the pendulum in the vertical position. After linearization, we have:

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \\ \delta\dot{x}_3 \\ \delta\dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{M - \frac{m_{pl}}{L}}\frac{K_m^2 K_g^2}{Rr^2} & -\frac{g m_{pl}}{L(M - \frac{m_{pl}}{L})} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{M(L - \frac{m_{pl}}{M})}\frac{K_m^2 K_g^2}{Rr^2} & \frac{g}{L - \frac{m_{pl}}{M}} & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M - \frac{m_{pl}}{L}}\frac{K_m K_g}{Rr} \\ 0 \\ -\frac{1}{M(L - \frac{m_{pl}}{M})}\frac{K_m K_g}{Rr} \end{bmatrix} V \tag{11}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \end{bmatrix} \tag{12}$$

By substituting values from Table 1, the linearized system yields:

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \\ \delta\dot{x}_3 \\ \delta\dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -15.14 & -3.04 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 37.23 & 31.61 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 3.39 \\ 0 \\ -8.33 \end{bmatrix} V \tag{13}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \end{bmatrix} \tag{14}$$

By achieving the linear model, we are now ready to design a controller to balance the inverted pendulum.

TABLE 1
SYSTEM PARAMETERS FOR CART/PENDULUM SYSTEM

Parameter	Symbol	Value	Units
Motor Torque Constant	K_m	0.00767	Nm/Amp
Gear-box Ratio	Kg	3.7	N/A
Motor Armature Resistance	R	2.6	Ω
Motor Pinion Radius	r	0.00635	m
Cart Mass	m_c	0.455	kg
Pendulum Mass	m_p	0.210	kg
Rotational Inertia	I	0.00651	$kg \times m^2$
Half-length of Pendulum	l	0.305	m

Step response for the system is shown below; as it can be seen, inverted pendulum has an unstable state at the commonplace. In this paper we proposed and compared two different methods with control the system.

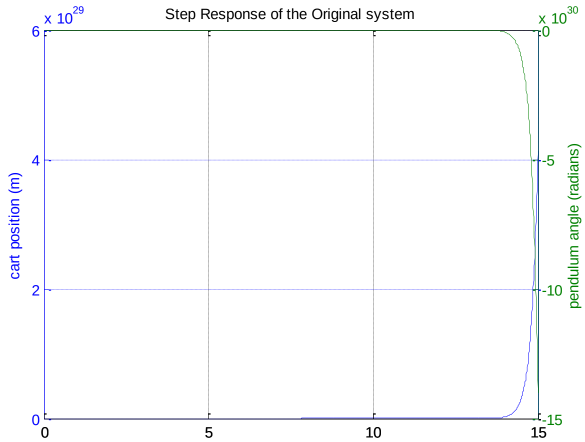

Fig.3- Step response of the inverted pendulum.

Before designing the controller, it is significant to verify that the system is controllable; in other words, is it possible to drive the state of the system anywhere we like. The system can be completely state controllable, if the controllability matrix has full rank where the rank of a matrix is the number of independent rows. The controllability matrix rank is 4 like the columns of the system. Since, it is possible to a design a proper controller for the system.

FULL STATE FEEDBACK (FSF)

The closed-loop input-output transfer function in state space can be characterized by the equation below:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}; \qquad (15)$$

$$y = C\underline{x} + D\underline{u}$$

with $x(t) \in R^N, u(t) \in R^N$. The initial condition is $x(0)$. Then the roots of the characteristic equation illustrate the poles of the system.

$$|sI - A| = 0 \qquad (16)$$

Full state feedback (FSF) is used by a command to the input vector ($\underline{u}$). Consider a matrix sense to the state vector:

$$\underline{u} = -K\underline{x} \qquad (17)$$

By substituting the equation above into the state space equations, it can be written as:

$$\dot{\underline{x}} = (A - BK)\underline{x}; \qquad (18)$$

$$y = (C - DK)\underline{x}$$

The equation, $\det(sI - (A - BK))$ characterizes the roots of the FSF system. By comparing the terms of this equation with the desired equation, the values of the feedback matrix K can be considered which forces the closed-loop eigen-values to the pole locations determined by the desired characteristic equation; because of that, this method is also known as pole placement method [5].

LINEAR QUADRATIC REGULATOR:

For the Linear Quadratic Regulator (with zero terminal cost), we set $\psi = 0$, and

$$L = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru, \qquad (19)$$

where the requirement that $L \geq 0$ implies that both Q and R are positive definite. For the linear plant dynamics also, we have

$$L_x = x^T Q \qquad (20)$$

$$L_u = u^T R$$

$$f_x = A$$

$$f_u = B$$

So that

$$x(t_0) = x_0 \qquad (21)$$

$$\dot{\lambda} = -Qx - A^T \lambda \qquad (22)$$

$$\lambda(t_f) = 0 \qquad (23)$$

$$Ru + B^T \lambda = 0 \qquad (24)$$

Therefore, the systems are perfectly linear, we try a connection $\lambda = Px$. Appending this into the $\dot{\lambda}$ equation, and then using the x equation, and by replacing for u, we achieve

$$PAx + A^T Px + Qx - PBR^{-1}B^T Px + \dot{P} = 0 \qquad (25)$$

This has to hold for all x; indeed, it is a matrix equation, the matrix Riccati equation [6]. The steady-state solution can be considered as:

$$PA + A^T P + Q - PBR^{-1} B^T P = 0 \quad (26)$$

This equation is the matrix algebraic Riccati equation (MARE), whose solution P is required to calculate the optimal feedback gain K [7]. The equation $Ru + B^T \lambda = 0$ gives the feedback law:

$$u = -R^{-1} B^T Px \quad (27)$$

The design steps to control system by LQR technique:

1. Select design parameter matrices Q and R
2. Solve the algebraic Riccati equation for P
3. Find the optimal value for the feedback by using $u = -R^{-1}B^TP$

SIMULATION AND RESULTS

State feedback control:
In this problem, the poles are characterized in:
P= {-55.7078,-3.4947 +10.5695i,-3.4947 -10.5695i,-5.3027 + 0.0000i}
Letting the considered gain k= [k1 k2 k3 k4] for controlling the state feedback, we have: $A - BK$ Comparing all the coefficient of above equation we found:

K=[-447.3448 -122.4159 -288.0091 -56.1644]

Step response for the controlled system is shown below:

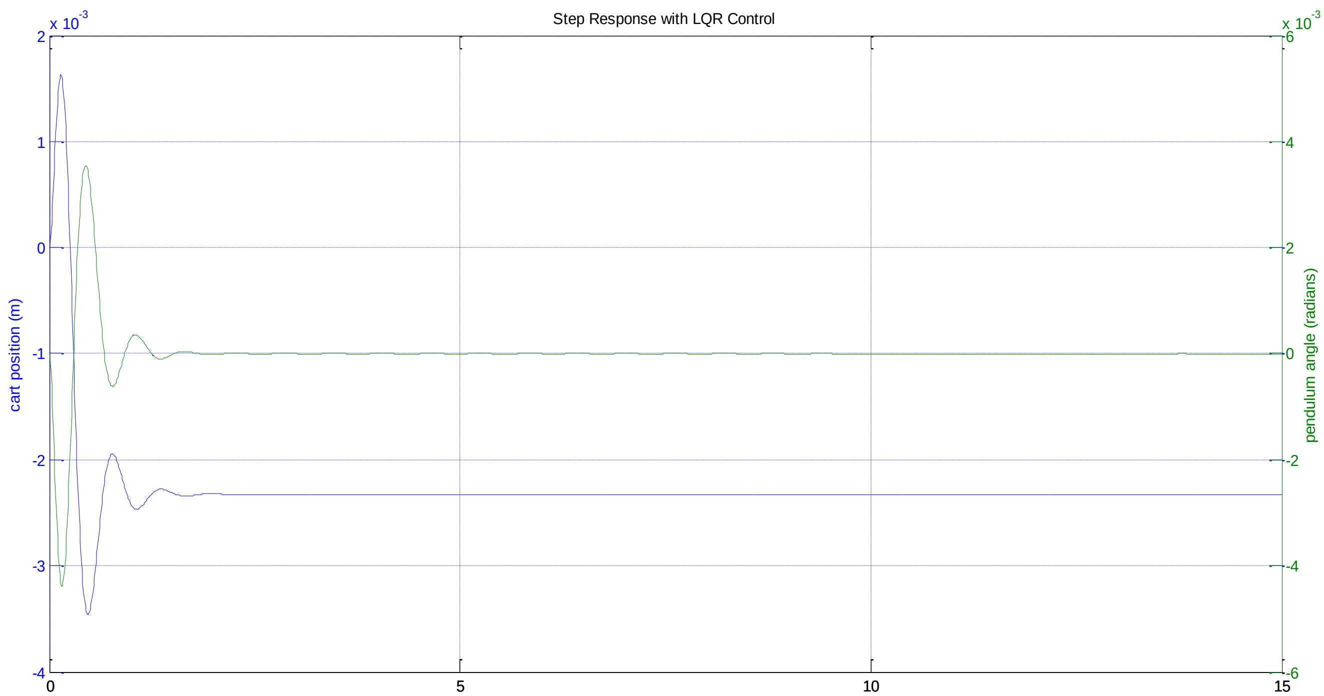

Fig.4-State space control response for the system

LQR Control:

By letting the $Q = C' \times C$:

Q=[1 0 0 0;0 0 0 0;0 0 1 0;0 0 0 0];

Different values for the R and the step response of the system are shown below:

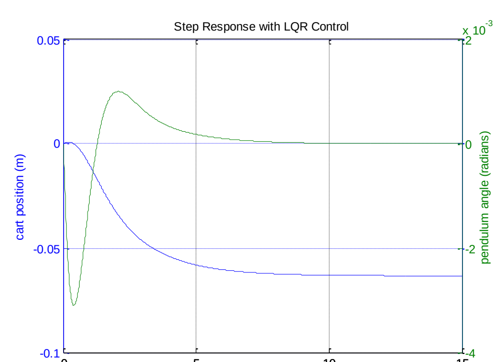

Fig.5-LQR response for R= 0.1

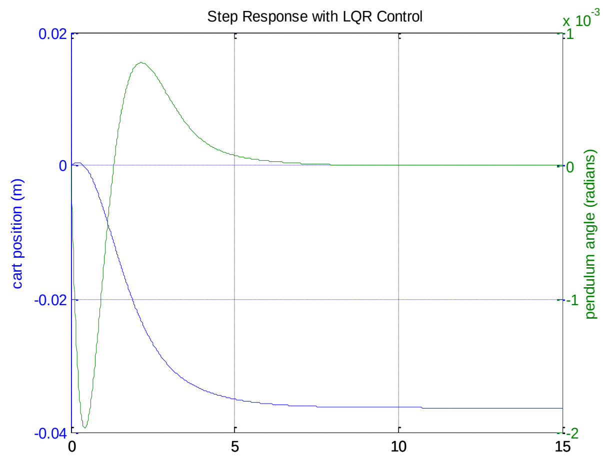

Fig.6-LQR response for R=0.033

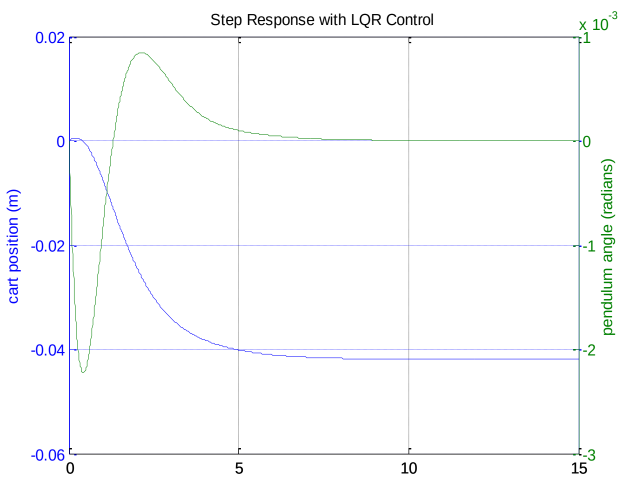

Fig.7-LQR response for R=0.044

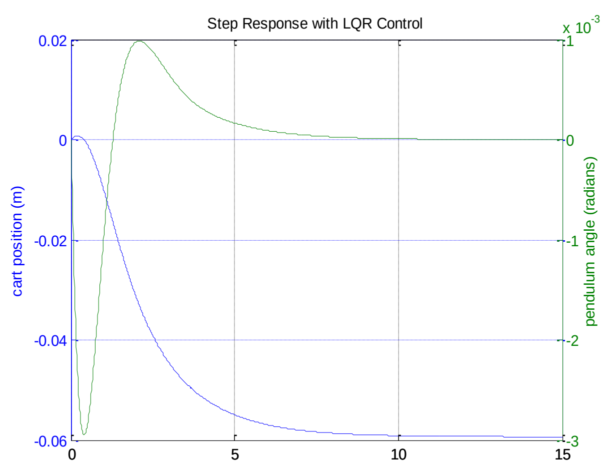

Fig.8-LQR response for R=0.088

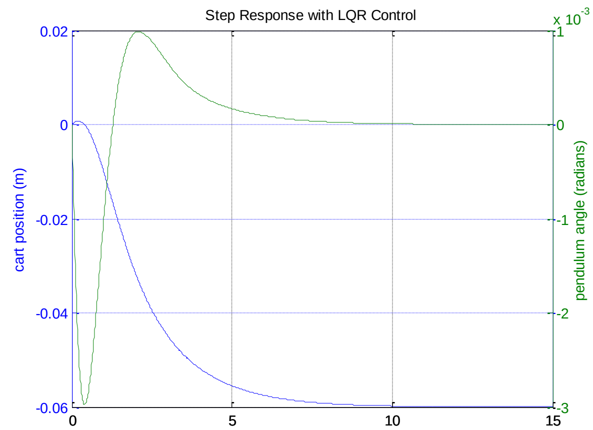

Fig.9-LQR response for R=0.090

As it can be seen from the above results, using LQR instead of FSF decreases the number of oscillations. Utilizing different values of R in LQR achieved different results. So it can be concluded that for having a good LQR controller, we have to select proper values of Q and R.

CONCLUSION

Two different control schemes have been implemented that will switch to a stabilizing controller when the pendulum is unbalanced. Inverted pendulum is one of the renowned unstable models to analyze the control techniques. Step response of the system is unstable with non-minimum phase zero. Applying state space feedback controller illustrated a stable state for the system. Provided LQR controller method resulted better results rather than the simple state feedback, but makes some troublous because of selection of constants of controller. Constant of the LQR controller can also be adjusted by the heuristic techniques for better results.

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